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One of the greatest engineering achievements of ancient times is a water tunnel, 1,036 meters (4,000 feet) long, excavated through a mountain on the Greek island of Samos in the sixth century B.C. It was dug through solid limestone by two separate teams advancing in a straight line from both ends, using only picks, hammers, and chisels. This was a prodigious feat of manual labor. The intellectual feat of determining the direction of tunneling was equally impressive. How did they do this? No one knows for sure, because no written records exist. When the tunnel was dug, the Greeks had no magnetic compass, no surveying instruments, no topographic maps, nor even much written mathematics at their disposal. Euclid's Elements, the first major compendium of ancient mathematics, was written some 200 years later.

There are, however, some convincing explanations, the oldest of which is based on a theoretical method devised by Hero of Alexandria five centuries after the tunnel was completed. It calls for a series of right-angled traverses around the mountain beginning at one entrance of the proposed tunnel and ending at the other, maintaining a constant elevation, as suggested by the diagram below left. By measuring the net distance traveled in each of two perpendicular directions, the lengths of two legs of a right triangle are determined, and the hypotenuse of the triangle is the proposed line of the tunnel. By laying out smaller similar right triangles at each entrance, markers can be used by each crew to determine the direction for tunneling. Later in this article I will apply Hero's method to the terrain on Samos.

Hero's plan was widely accepted for nearly 2,000 years as the method used on Samos until two British historians of science visited the site in 1958, saw that the terrain would have made this method unfeasible, and suggested an alternative of their own. In 1993, I visited Samos myself to investigate the pros and cons of these two methods for a Project MATHEMATICS! video program, and realized that the engineering problem actually consists of two parts. First, two entry points have to be determined at the same elevation above sea level; and second, the direction for tunneling between these points must be established. I will describe possible solutions for each part; but first, some historical background.
572 B.C. Pythagoras spent little of his adult life in Samos, and there is no reason to believe that he played a role in designing the tunnel.

Polycrates had a stranglehold on all coastal trade passing through the Strait of Mycale, and by 525 B.C., he was master of the eastern Aegean. His city was made virtually impregnable by a ring of fortifications that rose over the top of the 900-foot Mount Castro. The massive walls had an overall length of 3.9 miles and are among the best-preserved in Greece. To protect his ships from the southeast wind, Polycrates built a huge breakwater to form an artificial harbor. To honor Hera, queen of the Olympian gods, he constructed a magnificent temple that was supported by 150 columns, each more than 20 meters tall. And to provide his city with a secure water supply, he carved a tunnel, more than one kilometer long, two meters wide, and two meters high, straight through the heart of Mount Castro.

Delivering fresh water to growing populations has been an ongoing problem since ancient times. There was a copious spring at a hamlet, now known as Agiades, in a fertile valley northwest of the city, but access to this was blocked by Mount Castro. Water could have been brought around the mountain by an aqueduct, as the Romans were to do centuries later from a different source, but, aware of the dangers of having a watercourse exposed to an enemy for even part of its length, Polycrates ordered a delivery system that was to be completely subterranean. He employed a remarkable Greek engineer, Eupalinos of Megara, who designed an ingenious system. The water was brought from its source at Agiades to the northern mouth of the tunnel by an underground conduit that followed an 850-meter sinuous course along the contours of the valley, passing under three creek beds en route. Once inside the tunnel, whose floor was level, the water flowed in a sloping rectangular channel excavated along the eastern edge of the floor. The water channel then left the tunnel a few meters north of the southern entrance and headed east in an underground conduit leading to the ancient city. As with the northern conduit, regular inspection shafts trace its path.

The tunnel of Samos was neither the first nor the last to be excavated from both ends. Two other famous examples are the much shorter, and very sinuous, Tunnel of Hezekiah (also known as the Siloam tunnel), excavated below Jerusalem.
around 700 B.C., and a much longer tunnel under the English Channel completed in 1994. The Tunnel of Hezekiah required no mathematics at all (it probably followed the route of an underground watercourse), the Tunnel of Samos used very little mathematics, while the Channel Tunnel used the full power of modern technology. There is no written record naming the engineers for Hezekiah's tunnel, just as there is none for the pyramids of Egypt, most cathedrals of Europe, or most dams and bridges of the modern world. Eupalinos was the first hydraulic engineer whose name has been preserved. Armed only with intellectual tools, he pulled off one of the finest engineering achievements of ancient times. No one knows exactly how he did it. But there are several possible explanations, and we begin with Hero's method.

Hero, who lived in Roman Alexandria in the first century A.D., founded the first organized school of engineering and produced a technical encyclopedia describing early inventions, together with clever mathematical shortcuts. One of history's most ingenious engineers and applied mathematicians, Hero devised a theoretical method for aligning a level tunnel to be drilled through a mountain from both ends. The contour map on the right shows how his method could be applied to the terrain on Samos. This map is part of a detailed, comprehensive 250-page report published in 1995 by Hermann Kienast, of the German Archaeological Institute in Athens. The sloping dashed line on the map shows the tunnel direction to be determined.

Using Hero's method, start at a convenient point near the northern entrance of the tunnel, and traverse the western face of the mountain along a piecewise rectangular path (indicated in red) at a constant elevation above sea level, until reaching another convenient point near the southern entrance. Measure the total distance moved west, then subtract it from the total distance moved east, to determine one leg of a right triangle, shown dashed on the map, whose hypotenuse is along the proposed line of the tunnel. Then add the lengths of the north-south segments to calculate the length of the other leg, also shown dashed. Once the lengths of the two legs are known, even though they are buried beneath the mountain, one can lay out smaller horizontal right triangles on the terrain to the north and to the south (shown in orange) having the same shape as the large triangle, with all three hypotenuses on the same line. Therefore, workers can always look back to markers along this line to make sure they are digging in the right direction.

This remarkably simple and straightforward method has great appeal as a theoretical exercise. But to apply it in practice, two independent tasks need to be carried out with great accuracy: (a) maintain a constant elevation while going around the mountain; and (b) determine a right
angle when changing directions. Hero suggests doing both with a dioptra, a primitive instrument used for leveling and for measuring right angles. His explanation, including the use of the dioptra, was widely accepted for almost two millennia as the method used by Eupalinos. It was publicized by such distinguished science historians as B. L. van der Waerden and Giorgio de Santillana. But because there is no evidence to indicate that the dioptra existed as early as the sixth century B.C., other scholars do not believe this method was used by Eupalinos. To check the feasibility of Hero’s method, we have to separate the problems of right angles and leveling.

First, consider the problem of right angles. The Samians of that era could construct right angles, as evidenced by the huge rectangular stones in the beautifully preserved walls that extended nearly four miles around the ancient capital. Dozens of right angles were also used in building the huge temple of Hera just a few miles away. We don’t know exactly how they determined right angles, but possibly they constructed a portable rectangular frame with diagonals of equal length to ensure perpendicularity at the corners. A carpenter’s square appears in a mural on a tomb at Thebes, dating from about 1450 B.C., so it’s reasonable to assume that the Samians had tools for constructing right angles, although the accuracy of these tools is uncertain. In practice, each application of such a tool (the dioptra included) necessarily introduces an error of at least 0.1 degree in the process of physically marking the terrain. The schematic diagram on page 33 shows a level path with 28 right angles that lines up perfectly on paper, but in practice would produce a total angular error of at least two degrees. This would put the two crews at least 30 meters apart at the proposed junction. Even worse, several of these right angles would have to be supported by pillars 10 meters high to maintain constant elevation, which is unrealistic. A level path with pillars no more than one meter high would require hundreds of right angles, and would result in huge errors in alignment. Therefore, because of unavoidable errors in marking right angles, Hero’s method is not accurate enough to properly align the small right triangles at the two entrances.

As for leveling, one of the architects of the temple of Hera was a Samian named Theodoros, who invented a primitive but accurate leveling instrument using water enclosed in a rectangular clay gutter. Beautifully designed round clay pipes from that era have been found in the underground conduit outside the tunnel, and open rectangular clay gutters in the water channel inside the tunnel. So the Samians had the capability to construct clay gutters for leveling, and they could have used clay L-shaped pieces for joining the gutters at right angles, as suggested in the illustration on the left. With an ample supply of limestone slabs available, and a few skilled stonemasons, Eupalinos could have marked the path with a series of layered stone pillars capped by leveling gutters that maintained a constant elevation while going around the mountain, thereby verifying constant elevation with considerable accuracy.

In 1958, and again in 1961, two British historians of science, June Goodfield and Stephen Toulmin, visited the tunnel to check the practicability of Hero’s theory. They studied the layout of the surrounding countryside and concluded that it would have been extremely laborious—if not actually impossible—to carry out Hero’s method along the 55-meter contour line that joins the two entrances, because of ravines and overhangs. They also noticed that the tunnel was built under the only part of the mountain that could be climbed easily from the south, even though this placed it further from the city center, and they suggested that an alternative method had been used that went over the top, as shown in the photo above.

Armed with Goodfield and Toulmin’s analysis, I checked the feasibility of Hero’s method. It’s true that the terrain following the 55-meter contour is quite rough, especially at the western face of the mountain. But just a few meters below, near the 45-meter contour, the ground is fairly smooth, and it is easy to follow a goat trail through the brambles in a westerly direction around the mountain. Eupalinos could have cleared a suitable path along this terrain and marked it with stone pillars, keeping them at a constant elevation with clay leveling gutters, as described earlier, or with some other leveling instrument. At the western end of the south face, the terrain gradually slopes down into a stream-
bed that is usually dry. An easy walk along this streambed leads to the northern face and a gentle slope up toward the northern entrance of the tunnel. Not only is the path around the mountain almost level near the 45-meter contour, it is also quite short—from one tunnel entrance to the other is at most only 2,200 meters.

Goodfield and Toulmin suggested that the most natural way to establish a line of constant direction would be along the top of the mountain, driving a line of posts into the ground up one face of the hill, across the top, and down the other, aligning the posts by eye. To compare elevations on the two sides of the hill, they suggested measuring the base of each post against that immediately below it, using a level. This approach presents new problems. First, it is difficult to drill holes in the rocky surface to install a large number of wooden posts. Second, aligning several hundred posts by eye on a hillside is less accurate than Hero’s use of right angles. Keeping track of differences in elevation is also very difficult. There is a greater chance of error in measuring many changes in elevation along the face of a hill than there would be in sighting horizontally going around on a path of constant elevation. Because errors can accumulate when making a large number of measurements, Eupalinos must have realized that going over the top with a line of posts would not give a reliable comparison of altitude.

To ensure success, he knew it was essential for the two crews to dig along a nearly level line joining the two entrances. The completed tunnel shows that he did indeed establish such a line, with a difference of only 60 centimeters in elevation at the junction of the north and south tunnels, so he must have used a leveling method with little margin for error. Water leveling with clay gutters, as described earlier, provides one accurate method, but there are other methods that are easier to carry out in practice.

Recently, my colleague Mamikon Mnatsakanian conceived an idea for another simple leveling tool that could have been used, a long wooden bow suspended by a rope from a central balancing point so that its ends are at the same horizontal level. The bow need not have uniform thickness and could be assembled by binding together two lengthy shoots from, say, an olive tree. Such a bow, about eight meters long, would weigh about four pounds and would be easy to carry. And it has the advantage that no prior calibration is needed.

To use the bow as a leveling tool, place it on the ground and slowly lift it with the rope. If the two ends leave the ground simultaneously, they are at the same elevation. If one end leaves the ground first, place enough soil or flat rock beneath that end until the other end becomes airborne. At that instant, the bow will oscillate slightly, which can be detected visually, and the two ends will be at the same elevation. Three people are needed, one at the center and one at each end to fix the level points. To check the accuracy, turn the bow end-to-end and make sure both ends touch at the same two marks. In this way, the device permits self-checking and fine tuning. There will be an error in leveling due to a tiny gap between the endpoint of the bow and the point on which it is supposed to rest when level; this can be checked...
by the human eye at close range, and would be one or two millimeters per bow length.

To employ the leveling bow to traverse Mount Castro at constant elevation, one can proceed as follows. Construct two stone pillars, eight meters apart, and use the leveling bow to ensure that the tops of the pillars are at the same elevation. At the top of each pillar, a horizontal straightedge can be used for visual sighting to locate other points at the same elevation and on the same line—the same principle as used for aiming along the barrel of a rifle.

Points approximately at the same elevation can be sighted visually at great distances. Choose such a target point and construct a new pillar there with its top at the same elevation, and then construct another pillar eight meters away to sight in a new direction, using the leveling bow to maintain constant elevation. Continuing in this manner will produce a polygonal path of constant elevation. The contour map on the right shows such a path in green, with only six changes in direction leading from point $B$ near the northern entrance to point $D$ near the southern entrance. The distance around Mount Castro along this path is less than 2,200 meters (275 bow lengths), so an error of two millimeters per bow length could give a total leveling error of about 55 centimeters. This is close to the actual 60-centimeter difference in floor elevation measured at the junction inside the tunnel. In contrast to Hero’s method, no angular or linear measurements are needed for leveling with Mamikon’s bow.

Without Hero’s method, how could Eupalinos locate the entry points and fix the direction for tunneling? Kienast’s report offers no convincing theory about this, but using information in the report, Mamikon and I propose another method that could have determined both entries, and the direction of the line between them, with considerable accuracy. Kienast’s contour map (right)
shows that the north entrance to the tunnel, point A on the map, lies on the 55-meter contour line, and the source at Agiades is near the same contour line. His profile map in the report, which I have adapted for the diagram above, reveals that persons standing at the crest of the mountain cannot see Agiades in the north and the seashore in the south at the same time—but they could if they were on top of a seven-meter-high tower at point T. The Samians could easily have constructed such a tower from wood or stone.

Mamikon has conceived a sighting tool, based on the same principle as the Roman groma, that could have been used to align T with two points: one (N) near the source in the northern valley and one (S) on the seashore in the south within the city walls. This sighting tool consists of a pair of two-meter-high “fishing poles” about five meters apart, with a thin string hanging vertically from the top of each pole to which a weight or plumb bob has been attached. Each pole would be mounted on the tower so that it could turn around a vertical axis, allowing the hanging strings to be aligned by eye while aiming toward a person or marker at point N. Sighting along the same device in the opposite direction locates S.

Using the same type of sighting instrument from point N, and sighting back toward T, the tunnel’s northern entrance (point A) can be chosen so that S, T, N, and A are in the same vertical plane. Then the leveling bow can be used to ensure that A is also at the same elevation as N.

For the southern entrance, we need a point C inside the city walls at exactly the same elevation as A. Starting from a nearby point B at a lower elevation easier to traverse and in the vertical plane through N and A, a series of sighting pillars can be constructed around the western side of the mountain—using any of the leveling methods described earlier—until point D, at the intersection of the sighting plane through T and S, is reached. The two terminal points B and D will be at the same elevation above sea level, and the vertical plane through points B, T, and D fixes the line for tunneling. It is then a simple matter to sight a level line from a pillar at D to determine the southern entrance point, C, that has the same elevation as A. Markers could be placed along a horizontal line of sight at each entrance to guide the direction of excavation.

How accurate is this method of alignment? In sighting from one fishing pole to its neighbor five meters away, there is an alignment error of one millimeter, which expands to an overall error of 22 centimeters in sighting from point T at the top of the mountain to point N, 1,060 meters away to the north, plus another error of 13 centimeters from T to S, 644 meters away to the south. So the total error in sighting from N to S by way of T is of the order of 35 centimeters. Adjusting this to account for the alignment error from N to A by way of B, and from S to C by way of D, gives a total error of the order of less than 50 centimeters between the two directions of tunneling. This is well within the accuracy required to dig two shafts, each two meters wide, and have them meet, and is much more accurate than the other proposed methods of alignment.

Moreover, as with the alternative method of leveling, no angular or linear measurements are needed.
This picture was taken inside the tunnel looking toward the southern entrance. Although the floor is more or less level along the whole length, the roof slopes in line with the rock strata. Metal grillwork now covers the water channel on the left to prevent hapless tourists from falling in; the photo on the next page shows how treacherous it was before.

No matter how it was planned, the tunnel excavation itself was a remarkable accomplishment. Did the two crews meet as planned? Not quite. If the diggers had kept faith in geometry and continued along the straight-line paths on which they started, they would have made a nearly perfect juncture. The path from the north, however, deviates from a straight line. When the northern crew was nearly halfway to the junction point, they started to zigzag as shown in the diagram on the left, changing directions several times before finally making a sharp left turn to the junction point. Why did they change course? No one knows for sure. Perhaps to avoid the possibility of digging two parallel shafts. If one shaft zigzagged while the other continued in a straight line, intersection would be more likely. Or they may have detoured around places where water seeped in, or around pockets of soft material that would not support the ceiling. In the final stretch, when the two crews were near enough to begin hearing each other, both crews changed direction as needed and came together. The sharp turns and the difference in floor levels at the junction prove conclusively that the tunnel was excavated from both ends. At the junction itself, the floor level drops 60 centimeters from north to south, a discrepancy of less than one-eighth of a percent of the distance excavated. This represents an engineering achievement of the first magnitude.

A visit to the tunnel today reveals its full magnificence. Except for some minor irregularities, the southern half is remarkably straight. The craftsmanship is truly impressive, both for its precision and its high quality. The tunnel’s two-meter height and width allowed workers carrying rubble to pass those returning for more. The ceiling and walls are naked rock that gives the appearance of having been peeled off in layers. Water drips through the ceiling in many places and trickles down the walls, leaving a glossy, translucent coating of calcium carbonate, but some of the original chisel marks are still visible. The floor is remarkably level, which indicates that Eupalinos took great care to make sure the two entrances were at the same elevation above sea level.

Of course, a level tunnel cannot be used to deliver a useful supply of water. The water itself was carried in a sloping, rectangular channel excavated adjacent to the tunnel floor along its eastern edge. Carving this inner channel with hand labor in solid rock was another incredible achievement, considering the fact that the walls of the channel are barely wide enough for one person to stand in. Yet they are carved with great care, maintaining a constant width throughout. At the northern end, the bottom of the channel is about three meters lower than the tunnel floor, and it gradually slopes down to more than nine meters lower at the southern end. The bottom of the channel was lined with open-topped rectangular clay gutters like those in the drawing on page 34.
Jane Apostol descends the dauntingly narrow wooden staircase near the southern entrance, far left, which leads to a narrow walled passageway whose gabled roof is typical of ancient Greek construction, near left. Below: This old photo of the tunnel, looking north, shows the open water channel. Part of the channel above the water gutters was originally filled in with rubble, some of which is still visible.

When the tunnel was rediscovered in the latter part of the 19th century and partially restored, portions of the water channel were found to be covered with stone slabs, located two to three meters above the bottom of the channel, onto which excess rubble from the excavation had been placed up to the tunnel floor. The generous space below the slabs along the bottom of the channel provided a conduit large enough for ample water to flow through, and also permitted a person to enter for inspection or repairs. In several places the channel was completely exposed all the way up to the tunnel floor, permitting inspection without entering the channel. Today, most of the rubble has been cleared from the southern half of the channel, and metal grillwork installed to prevent visitors from falling in.

To visit the tunnel today, you first enter a small stone building erected in 1882 at the southern entrance. A narrow, rectangular opening in the floor contains a steep flight of wooden steps leading down into a walled passageway about 12 meters long, slightly curved, and barely wide enough for one person to walk through. The sides of the passageway are built of stone blocks joined without mortar, and capped by a gabled roof formed by pairs of huge, flat stones leaning against each other in a manner characteristic of ancient Greek construction. This passageway was constructed after the tunnel had been completely excavated. Excess rubble from the excavation, which must have been considerable, was used to cover the roof of this passageway, and a similar one at the north entrance, to camouflage them for protection against enemies or unwelcome intruders. Electric lights now illuminate the southern half, up to a point about 100 meters north of the junction with the northern tunnel, where there is a barrier and landfall that prohibits visitors from going further.
Modern stone walls now line the path near the northern entrance of the tunnel, left, while inside, right, a stone staircase leads down to the tunnel.

For centuries, the tunnel kept its secret. Its existence was known, but for a long time the exact whereabouts remained undiscovered. The earliest direct reference appears in the works of Herodotus, written a full century after construction was completed. Artifacts found in the tunnel indicate that the Romans had entered it, and there is a small shrine near the center from the Byzantine era (ca. A.D. 500–900). In 1853, French archaeologist Victor Guérin excavated part of the northern end of the subterranean conduit, but stopped before reaching the tunnel itself. An abbot from a nearby monastery later discovered the northern entrance and persuaded the ruler of the island to excavate it. In 1882, 50 men restored the southern half of the tunnel, the entire northern underground conduit, and a portion of the southern conduit. On the foundation of an ancient structure, they built a small stone house that today marks the southern entrance. In 1883, Ernst Fabricius of the German Archaeological Institute in Athens surveyed the tunnel and published an excellent description, including the topographic sketch shown on page 30.

After that, the tunnel was again neglected for nearly a century, until the Greek government cleared the southern half, covered the water channel with protective grillwork, and installed lighting so tourists could visit it safely. Today, the tunnel is indeed a popular tourist attraction, and a paved road from Pythagorion leads to the stone building at the southern entrance. Some visitors who enter this building go no further because they are intimidated by the steep, narrow, and unlit wooden staircase leading to the lower depths. At the north entrance, which can be reached by hiking around Mount Castro from the south, modern stone walls lead to a staircase and passageway. A portion of the northern tunnel has been cleared, but not lighted. The tunnel, the seawall protecting the harbor, and the nearby temple of Hera—three outstanding engineering achievements—were constructed more than 2,500 years ago. Today, the tunnel and seawall survive, and although the temple lies in ruins, a single column still stands as a silent tribute to the spirit and ingenuity of the ancient Greeks.

This lone survivor is one of 150 columns, each more than 20 meters tall, that once supported the Temple of Hera, lauded by Herodotus as one of the three greatest engineering feats of ancient times. The other two were also on Samos—the breakwater and the tunnel.

The videotape on the Tunnel of Samos has been a favorite of the educational Project MATHEMATICS! series (www.projectmathematics.com) created, directed, and produced by Professor of Mathematics, Emeritus, Tom Apostol. He is, however, better known to students both here and abroad as the author of textbooks in calculus, analysis, and number theory, books that have had a strong influence on an entire generation of mathematicians. After taking a BS (1944) and MS (1946) at the University of Washington, he moved to UC Berkeley for his PhD (1948), then spent a year each at Berkeley and MIT before joining Caltech in 1950. Although he became emeritus in 1992, his mathematical productivity has not slowed down—he has published 40 papers since 1990, 16 of them jointly with Mamikon Mnatsakanian.